### GEOMETRY.

**510. Proposed by JOSEPH E. ROWE, State College, Pa.**

Show how to find the equation of a line parallel to a side of the triangle of reference and passing through a given point, in any system of homogeneous coördinates, using the condition that two lines are parallel in this system but not the condition that two lines are perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of one escribed circle of the triangle.

**511. Proposed by FRANK V. MORLEY, Student, Haverford College, Pa.**

Let $a_i$ $(i = 1, 2, 3, 4)$ be four points on a circle and let the in-center of the triangle formed by omitting $a_i$ be $c_i$; prove that the four points $c_i$ form a rectangle.

### CALCULUS.

**425. Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.**

Show that the infinite product

$$(1 - z)(1 + \tfrac{1}{2}z)(1 - \tfrac{1}{3}z)(1 + \tfrac{1}{4}z)\cdots = \frac{\sqrt{\pi}}{\Gamma(1 + \frac{1}{2}z)(\frac{1}{2} - \frac{1}{2}z)}.$$

**426. Proposed by C. N. SCHMALL, New York City.**

If $A$ be the area of a plane triangle constructed with the sides $a$, $b$, $c$, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of $A$ is $\frac{1}{4}k^2$.

### MECHANICS.

**342. Proposed by WILLIAM HOOVER, Columbus, Ohio.**

A uniform rod of length $2a$ is freely hinged at one end, at the other end a string of length $b$ is attached which is fastened at its further end to a point on the surface of a homogeneous sphere of radius $c$. If the masses of the rod and sphere are equal, find the motion of the system when slightly disturbed from the vertical, and the cubic equation giving the corresponding small oscillations.

**343. Proposed by J. ROSENBAUM, New Haven, Conn.**

Two bodies of equal masses and coefficients of friction $\mu_1$ and $\mu_2$ are connected by a light spring of stiffness $k$ and placed on an inclined plane. Discuss the motion of each body when the angle between the non-stretched spring and the plane is $\theta$.

### NUMBER THEORY.

**261. Proposed by NORMAN ANNING, Chilliwack, B. C.**

Show that for any positive integer $n$ (excluding powers of 2) positive integers $a_1$, $a_2$, $a_3$, $\cdots a_k$, which are less than $n/2$ can be chosen in such a way that

$$2^k \cos (a_1\pi/n) \cos (a_2\pi/n) \cos (a_3\pi/n) \cdots \cos (a_k\pi/n) = 1.$$

**262. Proposed by C. N. SCHMALL, New York City.**

If $x$, $y$, $z$ are three integers, consecutive among the integers prime to 3, show that

$$x(x - 2y) - z(z - 2y) = \pm 3.$$

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

J. A. BULLARD and J. W. BALDWIN solved 464. These solutions were received after selections for publication were made.

**465. Proposed by CYRUS B. HALDEMAN, Ross, Ohio.**

Having given $\tan^{-1} 1 = \tan^{-1} 1/2 + \tan^{-1} 1/3$, show that

$$\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57.$$

## SOLUTION BY HORACE OLSON, Chicago, Illinois.

Put $\tan^{-1} 1/2 = \tan^{-1} 1/3 + \tan^{-1} x_1 = \tan^{-1}\dfrac{3x_1 + 1}{3 - x_1}$. From this equation $x_1$ is found to be 1/7. Hence, $\tan^{-1} 1 = 2 \tan^{-1} 1/3 + \tan^{-1} 1/7$.

Now put

$$\tan^{-1} 1/3 = \tan^{-1} 1/7 + \tan^{-1} x_2 = \tan^{-1}\frac{7x_2 + 1}{7 - x_2};$$

whence $x_2 = 2/11$, and $\tan^{-1} 1 = 3 \tan^{-1} 1/7 + 2 \tan^{-1} 2/11$.

Now put

$$\tan^{-1} 1/7 = \tan^{-1} x_3 + \tan^{-1} y_3 = \tan^{-1}\frac{x_3 + y_3}{1 - x_3y_3}.$$

This gives the indeterminate equation $x_3y_3 + 7(x_3 + y_3) = 1$, or $(x_3 + 7)(y_3 + 7) = 50$. One set of solutions of this equation is $x_3 = 1/8$, $y_3 = 1/57$. Hence, $\tan^{-1} 1 = 3 \tan^{-1} 1/8 + 3 \tan^{-1} 1/57 + 2 \tan^{-1} 2/11$. Put

$$\tan^{-1} 2/11 = \tan^{-1} 1/8 + \tan^{-1} x_4 = \tan^{-1}\frac{8x_4 + 1}{8 - x_4};$$

whence $x_4 = 1/18$. Hence, $\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57$.

Also solved by E. B. ESCOTT, W. J. THOME, M. T. REED, G. W. HARTWELL, and E. E. WHITEFORD.

**466. Proposed by E. B. ESCOTT, Kansas City, Mo.**

For what functions, $f$, are the following relations true:

When

$$\frac{f(x, y, z)}{X} = \frac{f(y, z, x)}{Y} = \frac{f(z, x, y)}{Z},$$

then

$$\frac{f(X, Y, Z)}{x} = \frac{f(Y, Z, X)}{y} = \frac{f(Z, X, Y)}{z}?$$

## SOLUTION BY ALBERT A. BENNETT, University of Texas.

We notice that $f(X, Y, Z)$ must be homogeneous, since if we replace $X, Y, Z$ by $cX, cY, cZ$, in the first relation it is unaltered, and hence also the second. Thus $x, y, z$ may be regarded as the homogeneous coördinates of the points in one plane, $X, Y, Z$ of those in a second. The problem may therefore be expressed as follows: What are the involutoric plane transformations with triangular symmetry? Here "involutoric" is used in the restricted sense as of period two, and triangular symmetry is used in approximately the sense first suggested by Clifford, the point (1, 1, 1) being a center of triangular symmetry. There can be little doubt that there exist transformations of this form which are essentially transcendental and which may be regarded as limiting cases of algebraic birational transformations. To catalogue the explicit forms of even the algebraic cases is perhaps out of the question, since no explicit classification of algebraic forms of Cremona transformations has been attempted beyond the very simplest cases. Normal forms under the Cremona group are indeed known. Compare Pascal's *Repertorium* or other detailed articles on birational geometry.

It is furthermore obvious that a given geometric solution gives rise to an infinite number of analytic solutions. For example, if $g(x, y, z)$ be one solution, and $\varphi(x, y, z)$ be a constant or the equation of any triangularly symmetric self-corresponding curve whether the self-correspondence be singular or not, then $\varphi(x, y, z)g(x, y, z)$ is a solution. Apart from the trivial solutions $f(x, y, z) = 0$ and $f(x, y, z) = x$, the simplest case is probably the quadratic transformation given by $f(x, y, z) = 1/x$, and then taking $\varphi(x, y, z) = xyz$, or $\sqrt{xyz}$, or $xyz/\log(1 + xyz)$ or $(xy + yz + zx)(x + y + z)$, etc., we get other related solutions from $f(x, y, z) = 1/x$.